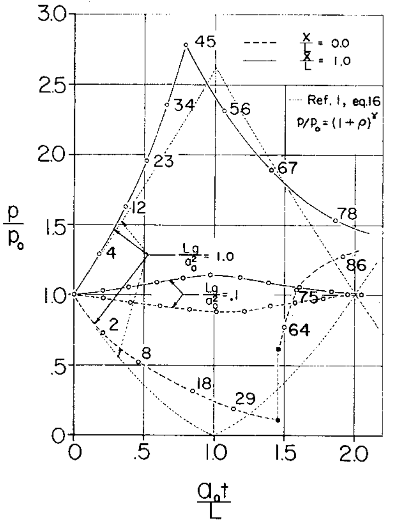

Fig. 3 Pressure vs nondimensionalized time.

slopes of the C^+ and C^- characteristics evaluated at points 1 and 2, respectively. The values of τ_{23}' and τ_{13}' are read directly from the physical plane (Fig. 1a). If we denote

$$\frac{2}{\gamma - 1}\frac{a}{a_0} + \frac{u}{a_0} = P$$

$$\frac{2}{\gamma - 1}\frac{a}{a_0} - \frac{u}{a_0} = Q$$

their graphic representations are straight lines in the state plane (Fig. 1b). Hence, Eq. (8) represents the change in the variables P and Q with respect to time; thus,

$$P_3 = P_1 + (\delta P/\delta\tau)\Delta\tau_{13} = P_1 + (L\alpha/a_0^2)|_{av}\Delta\tau_{13}$$

$$Q_3 = Q_2 + (\delta Q/\delta\tau)\Delta\tau_{23} = Q_2 - (L\alpha/a_0^2)|_{av}\Delta\tau_{23}$$

Since the value of $L\alpha/a_0^2$ is fixed, the first estimates of $P_3 - P_1 = \Delta_+$ and $Q_3 - Q_2 = \Delta_-$ can be made and point 3′ fixed in the state plane. A new estimate of point 3 is made in the physical plane, point 3″, and the iteration process continued until the values converge on point 3. The boundary condition that must be satisfied during the construction of the solution in the state and physical is that the gas velocity at both ends of the vessel must be zero.

The advantage of the graphical approach is that problems with very difficult boundary conditions can be solved with the same ease as the examples presented, whereas it may be very difficult if not impossible to obtain the inverse transform for the linearized equations of motion and apply the boundary conditions.

Two solutions have been carried out for a unit constant nondimensionalized acceleration, $L\alpha/a_0^2$, of 1 and 0.1 for a comparison with the linearized approach. The state and physical planes are only shown for the larger value of acceleration (Fig. 2). From the isentropic relationship of pressure vs speed of sound, the pressure variation at each end of the vessel is plotted in Fig. 3.

Analysis of Results

From Fig. 3, it is clearly seen that the pressure deviates significantly from the reference or initial value for the largest nondimensional acceleration. In addition, the pressure waves do not appear as simple harmonics, as would be expected from the linearized approach. A shock wave with a pressure ratio of about 2.4 is formed inside the vessel. At the lower acceleration rate, the deviations are much smaller; however, a definite shift of pressure vs time curve due to the nonlinearity of the equations of motion is still noticeable.

Conclusion

The pressure-time history due to the motion of a perfect gas in a cylindrical vessel of uniform cross-sectional area undergoing an arbitrary acceleration as a function of time in the direction of the major axis of the vessel can be obtained through the application of a graphical or numerical technique based on the method of characteristics. A comparison of the pressure-time curves for the system undergoing a unit constant acceleration indicates that the results based on linearizing the equations of motion[1] will be in close agreement with those based on the method of characteristics for a nondimensionalized acceleration, $L\alpha/a_0^2$, less than 0.1.

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Quadratic Effects of Frequency on Aerodynamic Derivatives

Kazimierz J. Orlik-Rückemann*
National Aeronautical Establishment, Ottawa, Ontario, Canada

STATIC and dynamic stability derivatives determined from oscillatory experiments in wind tunnels often display significant effects of reduced frequency. Although it is common practice to present such effects in graphical form, it is suggested here that often it may be preferable to use an analytical presentation, involving second- and third-order derivatives. The analysis of the equation of motion for a single-degree-of-freedom oscillatory system is presented here with emphasis on the effect of these additional derivatives in the general case of amplitude that varies with time. This new approach is illustrated with some examples, based on a recent wind-tunnel investigation of sweptback wings at supersonic speeds.

Equation of Motion

The following expressions are given in terms of variables and coefficients applicable to an oscillatory motion in pitch but, of course, can be adapted for any single-degree-of-freedom oscillation. The equation of motion for such a system can be written as

$$I\ddot{\theta} + \gamma\dot{\theta} + K\theta = M \tag{1}$$

where I is moment of inertia, γ is mechanical damping coefficient, K is mechanical stiffness coefficient, and θ, $\dot{\theta}$, and $\ddot{\theta}$ are the angle of oscillation in pitch around a fixed axis† and its first and second derivatives with respect to time, respectively. M is the aerodynamic pitching moment, which, in turn, can be expressed in terms of successive time derivatives of the angle of oscillation, viz.,

$$M = M_\theta\theta + M_{\dot{\theta}}\dot{\theta} + M_\sigma\ddot{\theta} + M_{\dot{\sigma}}\dddot{\theta} + \ldots \tag{2}$$

where $\dddot{\theta}$ is the third time derivative of the angle of oscillation, and M_θ, $M_{\dot{\theta}}$, M_σ, and $M_{\dot{\sigma}}$ are the successive pitching moment derivatives.‡

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* Head, Unsteady Aerodynamics Laboratory. Associate Fellow Member AIAA.

† Thus, compared to standard nomenclature, we have $M_\theta = M_\alpha$ but $M_{\dot{\theta}} = M_q + M_{\dot{\alpha}}$.

‡ The second- and third-order derivatives are denoted by M_σ and $M_{\dot{\sigma}}$ to avoid double and triple dotted subscripts.

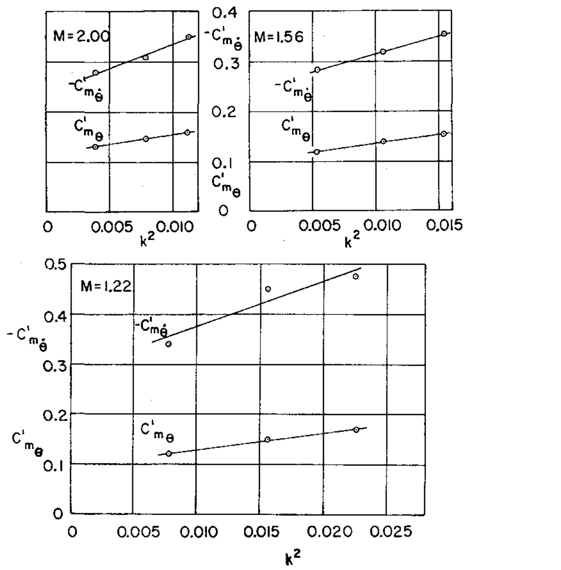

Fig. 1 Variation of $C'_{m\theta}$ and $C'_{m\dot\theta}$ with k^2 for aspect ratio 2 sweptback wing with leading edge sweep 60° and taper ratio 0.238, oscillating around an axis at 82.7% of root chord.

For the general case of amplitude varying with time, the instantaneous angle of oscillation can be written as

$$\theta = \theta_0 e^{(i\omega - \nu\delta)t} \tag{3}$$

where $\omega = 2\pi\nu$ is circular frequency, δ is logarithmic decrement, t is time, and θ_0 is the amplitude at $t = 0$.

The quantities $\ddot\theta$ and θ in Eq. (2) are replaced now by expressions involving only θ and $\dot\theta$, which can be derived by successive differentiation of Eq. (3). Neglecting terms of order $\nu^2\delta^2$, which are small compared to ω^2, Eq. (1) can then be rewritten as

$$I\ddot\theta + (\gamma - M_{\dot\theta} + 2\nu\delta M_\sigma + \omega^2 M_{\dot\sigma})\dot\theta + (K - M_\theta + \omega^2 M_\sigma - 2\omega^2\nu\delta M_{\dot\sigma})\theta = 0 \tag{4}$$

Second- and Third-Order Derivatives

Defining the dimensionless derivatives by the equation

$$C_m = \frac{2M}{\rho V^2 Sc} = \theta C_{m\theta} + \dot\theta\left(\frac{c}{2V}\right) C_{m\dot\theta} + \ddot\theta\left(\frac{c}{2V}\right)^2 C_{m\sigma} + \dddot\theta\left(\frac{c}{2V}\right)^3 C_{m\dot\sigma} \tag{5}$$

where ρ is density, V is velocity, S is reference area, and c is reference length, we obtain

$$C'_{m\theta} = C_{m\theta} - k^2 C_{m\sigma} + (k^3\delta/\pi)C_{m\dot\sigma} \tag{6}$$

$$C'_{m\dot\theta} = C_{m\dot\theta} - (k\delta/\pi)C_{m\sigma} - k^2 C_{m\dot\sigma} \tag{7}$$

where $k = \omega c/2V$ is reduced frequency. $C'_{m\theta}$ and $C'_{m\dot\theta}$ are generalized stiffness and damping derivatives, respectively, which can be obtained by any of the numerous data reduction methods for oscillatory experiments. Thus, for instance, for free oscillation experiments we have

$$C'_{m\theta} = -\frac{2I}{\rho V^2 Sc}(\omega^2 - \omega_0^2) \tag{8}$$

$$C'_{m\dot\theta} = -\frac{8I}{\rho V Sc^2}(\nu\delta - \nu_0\delta_0) \tag{9}$$

where subscript zero denotes quantities measured in vacuum.

The derivatives $C_{m\sigma}$ and $C_{m\dot\sigma}$ can be calculated from values of $C'_{m\theta}$ and $C'_{m\dot\theta}$ obtained at two reduced frequencies, say k_H and k_L, from the following expressions:

$$C_{m\sigma} = \frac{(k_L^2 - k_H^2)(C'_{m\theta H} - C'_{m\theta L}) + R(C'_{m\dot\theta H} - C'_{m\dot\theta L})}{(k_L^2 - k_H^2)^2 + PR} \tag{10}$$

$$C_{m\dot\sigma} = \frac{(k_L^2 - k_H^2)(C'_{m\dot\theta H} - C'_{m\dot\theta L}) - P(C'_{m\theta H} - C'_{m\theta L})}{(k_L^2 - k_H^2)^2 + PR} \tag{11}$$

where

$$P = (1/\pi)(k_L\delta_L - k_H\delta_H) \qquad R = (1/\pi)(k_L^3\delta_L - k_H^3\delta_H)$$

However, the terms involving k^3 often are small and can be neglected, thus reducing Eqs. (10) and (11) to

$$C_{m\sigma} = \frac{C'_{m\theta H} - C'_{m\theta L}}{k_L^2 - k_H^2} \tag{10a}$$

$$C_{m\dot\sigma} = \frac{C'_{m\dot\theta H} - C'_{m\dot\theta L} - PC_{m\sigma}}{k_L^2 - k_H^2} \tag{11a}$$

Having calculated $C_{m\sigma}$ and $C_{m\dot\sigma}$, the derivatives $C_{m\theta}$ and $C_{m\dot\theta}$ can now be obtained from Eqs. (6) and (7). The four derivatives thus determined constitute a basic set from which the generalized stiffness and damping derivatives can be calculated for any reduced frequency. If the terms involving δ are significant, the resulting value of δ must be determined by a trial and error procedure. Omission of those terms is equivalent to the assumption of constant amplitude oscillation.

Application to Experimental Data

The present analysis is based on the assumption of quadratic variation of the generalized derivatives $C'_{m\theta}$ and $C'_{m\dot\theta}$ with reduced frequency, which is expected to be adequate for many practical cases. That, however, should be checked carefully for any new type of configuration or new range of experimental conditions. If the effect of reduced frequency appears to be significantly different from quadratic, more terms in Eq. (2) may be required, making the analysis more complex or even rendering it impractical.

In Fig. 1 an example is given of the validity of the present approximation for a particular case of a thin sweptback wing performing small amplitude oscillation around zero mean incidence. The generalized derivatives $C'_{m\theta}$ and $C'_{m\dot\theta}$ are given as functions of k^2 for three supersonic Mach numbers. Although some deviations from the quadratic variation

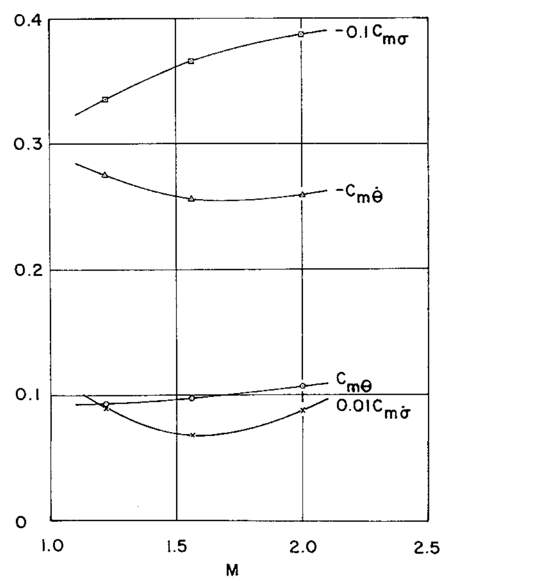

Fig. 2 Pitching moment derivatives for the same wing and axis of oscillation as in Fig. 1.

can be observed, the general effect of reduced frequency appears to be described adequately by the assumption made, with the possible exception of data obtained at the lowest Mach number (1.22). The four pitching moment derivatives, as determined by the present analysis, are shown as functions of Mach number in Fig. 2. All experimental data are taken from Ref. 1, where the questions of a similar representation of frequency effects on lift force derivatives and of the resulting axis transfer equations also are discussed.

Concluding Remarks

It is suggested that in cases when frequency effects are quadratic it may be convenient to describe them analytically by introducing second- and third-order derivatives. Such representation, of course, also is more easily adaptable for subsequent application in stability equations. The additional derivatives required can be determined from results of standard oscillatory experiments performed at two frequencies. In addition, at least one more frequency may be required for ascertaining the quadratic form of frequency effects, but these additional experiments can probably often be limited to one representative model at one or two sets of experimental conditions.

Frequency effects, of course, are of interest whenever information is sought which pertains to full-scale conditions featuring relatively high reduced frequencies. It is not equally obvious that frequency effects may also be important in the opposite case, namely when full-scale information involving low reduced frequencies is desired. This occurs when experimental results, because of facilities or techniques available, can only be obtained at high reduced frequency, as is the case, e.g., in short run-duration facilities such as hypersonic shock tunnels or hotshot tunnels. The present method of data analysis and presentation may be very convenient for extrapolating such results to lower frequencies.

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Similarity in Axisymmetric Viscous Free Mixing with Streamwise Pressure Gradient

M. H. Steiger*
Polytechnic Institute of Brooklyn, Freeport, N. Y.

A SURVEY of the literature on viscous free-mixing with streamwise pressure gradients has been given in Refs. 1 and 2. In the Introduction of Ref. 1, which emphasizes linearized flow, Steiger and Bloom have stated a nonlinear similarity equation for axisymmetric free-mixing with pressure gradients, admitting large velocity defects. They also point out that these types of axisymmetric solution have not yet been treated in the literature. This note presents a brief derivation of the axisymmetric similarity equation in more general form, and gives a particular solution (other than that of uniform flow) which may be expressed in closed form.

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* Assistant Professor. Member AIAA.

The following boundary-layer equations are assumed to govern the incompressible, axisymmetric viscous free-mixing:

Continuity

$$(ur)_x + (vr)_r = 0 \tag{1}$$

Momentum

$$uu_x + vu_r = (\nu/r)(ru_r)_r + u_e u_{ex} \tag{2}$$

where x and r are, respectively, the streamwise and radial coordinates with corresponding velocity components u and v, ν is the kinematic viscosity, subscripts x and r denote partial differentiation with respect to the indicated variable, and subscript e denotes inviscid conditions at the outer edge of the viscous layer.

The boundary conditions are, at

$$r = 0 \qquad u_r = 0 \qquad v = 0 \tag{3a}$$

$$u = u_e \quad \text{as} \quad r \to \infty \tag{3b}$$

Let

$$\eta = r/N(s) \tag{4a}$$

$$s = x \tag{4b}$$

and define a similarity parameter as follows:

$$u = u_e(s)[F'(\eta)/\eta] \tag{5}$$

Equation (1) is satisfied by introducing a stream function, such that $ur = \psi_r$ and $vr = -\psi_x$, and therefore, with Eqs. (4a) and (5), it follows that $\psi = u_e N^2 F$.

By utilizing well-known operations, Eq. (2) reduces to

$$\left[\eta\left(\frac{F'}{\eta}\right)'\right]' + \frac{1}{\nu}\frac{d(N^2u_e)}{ds} F\left(\frac{F'}{\eta}\right)' + \frac{N^2}{\nu}\frac{du_e}{ds}\eta\left[1 - \left(\frac{F'}{\eta}\right)^2\right] = 0 \tag{6}$$

Equation (6) is subject to the boundary conditions

$$\text{at } \eta = 0 \qquad F = 0 \qquad (F'/\eta)' = 0 \tag{7a}$$

and

$$\text{as } \eta = \infty \qquad (F'/\eta) = 1 \tag{7b}$$

The requirements for similarity are

$$(1/\nu)[d(N^2u_e)/ds] = \alpha = \text{const} \tag{8a}$$

$$(N^2/\nu)(du_e/ds) = \beta = \text{const} \tag{8b}$$

and

$$\text{at } \eta = 0 \qquad F'/\eta = \text{const} \tag{9}$$

Two variations of $u_e(x)$ can be derived from conditions (8). If α is nonzero, then, without loss in generality, it can be set equal to unity. With $\alpha = 1$, Eqs. (8a) and (8b), with (4b), yield, respectively,

$$N^2u_e = \nu(x - x_c) \qquad u_e = u_{e_c}(x/x_c)^\beta \tag{10}$$

where subscript c denotes conditions at an initial station.

If $\alpha = 0$, then Eq. (8a) gives

$$N^2 = \nu/u_e \tag{11a}$$

whereas (8b), with Eq. (4b), yields

$$u_e = u_{e_c}e^{\beta(x-x_c)} \tag{11b}$$

Condition (9), in all cases, implies that u_0/u_e is a constant. (Here u_0 is the streamwise velocity at $r = 0$.)

In general, in order to obtain solutions of the system (6) and (7), numerical methods are required. However, it can readily